Laplacian spectral characterization of some double starlike trees

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Abstract

A tree is called *double starlike* if it has exactly two vertices of degree greater than two. We denote by $H(p,n,q)$ one double starlike tree, which is obtained by attaching the centers of two stars $K_{1,p}$ and $K_{1,q}$ to the ends of a path P_n respectively. In this paper, graph $H(p,n,q)$ is proved to be determined by its Laplacian spectrum.

keywords: Laplacian spectrum; L-cospectral graphs; Line graph

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1 Introduction

We start with some standard conceptions of graphs, as introduced in most textbooks on graph theory (e.g. [1, 3]). Let $G = (V(G), E(G))$ be an undirected and simple graph (loops and multiple edges are not allowed) with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G)$, where $v_1, v_2, \dots, v_n$ are indexed in the non-increasing order of degrees. Let matrix $A(G)$ be the *(0,1)-adjacency matrix* of G and $d_i = d_i(G) = d_G(v_i)$ the degree of the vertex v_i. The matrix $L(G) = D(G) - A(G)$ is called the *Laplacian matrix* of G, where $D(G)$ is the $n \times n$ diagonal matrix with $\{d_1, d_2, \dots, d_n\}$ as diagonal entries. The eigenvalues of matrices $A(G)$ and $L(G)$ are called the *adjacency eigenvalues* and *Laplacian eigenvalues* of G, respectively. We denote by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n (= 0)$ the adjacency eigenvalues and the Laplacian eigenvalues of G, respectively. The multiset of eigenvalues of $A(G)$ (resp. $L(G)$) is called the *adjacency* (resp. *Laplacian*) *spectrum* of G. Two graphs are said to be A*-cospectral* (resp. L*-cospectral*) if they have the same adjacency (resp. Laplacian) spectrum.

A graph is said to be determined by its Laplacian (resp. adjacency) spectrum if there is no other non-isomorphic graphs L-cospectral (resp. A-cospectral) with it. To characterize some graphs by their spectra is very difficult. For its background, we refer the readers to [3–5].

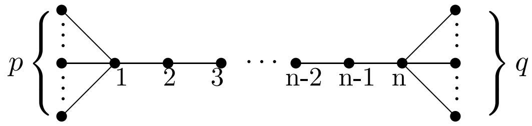

Fig. 1. Double starlike tree $H(p,n,q)$

Up until now, some trees with special structures have been proved to be determined by their spectra, for example, the path [4], the T-shape tree [20], and the starlike tree [17]. Moreover, one double starlike tree, denoted by $H_n(p,p)$ ($n \geq 2$, $p \geq 1$), was proved to be determined by its

Laplacian spectrum in [11]. In the Conclusion of [11], the authors posed the double starlike tree $H(p,n,q)$ ($n \geq 1$, $p \geq 1$, $q \geq 1$) (shown in Figure 1) and indicated the difficulties to prove that $H(p,n,q)$ is determined by its Laplacian spectrum. Indeed, if we let $p = q \geq 2$, then $H(p,n,p)$ is $H_n(p,p)$. Clearly, if $p = q = 1$, then $H(1,n,1)$ is a path of order $n+2$. If $n = 1$, then $H(p,1,q)$ is a star $K_{1,p+q}$ of order $p+q+1$. It is known that paths, stars and $H_n(p,p)$ are all determined by their Laplacian spectra [4, 11, 17]. Therefore, we will only need to consider $H(p,n,q)$ for $n \geq 2$, $p \neq q \geq 2$. Without loss of generality, we let $p > q \geq 2$. In [13–15, 19], the authors investigated double starlike trees $H(p,n,q)$ with $p > q = 2$ or $p - q = 1, 2, 7$ and proved that they were determined by their Laplacian spectra repectively. Unfortunately, they did not present the most powerful method to solve the problem of Laplacian spectral characterization of $H(p,n,q)$, although they solved it partially.

In this paper, we will solve this problem completely. We pose a new method for $H(p,n,q)$ with $n \geq 2$, $p > q \geq 2$. Then together with the result in [11], we can conclude that every double starlike tree $H(p,n,q)$ is determined by its Laplacian spectrum.

2 Preliminaries

In order to show our main result, some previously established results are summarized in this section.

Lemma 2.1. [6] *Let T be a tree with n vertices and $\mathcal{L}(T)$ be its line graph. Then for $i = 1, 2, \ldots, n-1$, $\mu_i(T) = \lambda_i(\mathcal{L}(T)) + 2$.*

Lemma 2.2. [12] *Let u be a vertex of G, and $G - u$ be the subgraph obtained from G by deleting u together with its incident edges. Then*

$$\mu_i(G) \geq \mu_i(G-u) \geq \mu_{i+1}(G) - 1, \quad i = 1, 2, \ldots, n-1.$$

Lemma 2.3. [4] *Suppose that N is a symmetric $n \times n$ matrix with eigenvalues $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$. Then the eigenvalues $\alpha'_1 \geq \alpha'_2 \geq \cdots \geq \alpha'_m$ of a principal submatrix of N of size m satisfy $\alpha_i \geq \alpha'_i \geq \alpha_{n-m+i}$ for $i = 1, 2, \ldots, m$.*

Lemma 2.4. [2] *In a simple graph, the number of closed walks of length 4 equals twice the number of edges plus four times of the number of paths on three vertices plus eight times of the number of 4-cycles.*

Lemma 2.5. [4, 16] *Let G be a graph. For the adjacency matrix and the Laplacian matrix, the following can be deduced from the spectrum.*
(1) *The number of vertices.*
(2) *The number of edges.*
For the adjacency matrix, the following follows from the spectrum.
(3) *The number of closed walks of any length.*
For the Laplacian matrix, the following follows from the spectrum.
(4) *The number of components.*
(5) *The number of spanning trees.*
(6) *The sum of the squares of degrees of vertices.*

Lemma 2.6. [8,9] *Let G be a graph with $V(G) \neq \emptyset$ and $E(G) \neq \emptyset$. Then*

$$d_1 + 1 \leq \mu_1 \leq \max\left\{\frac{d_i(d_i + m_i) + d_j(d_j + m_j)}{d_i + d_j}, \quad v_iv_j \in E(G)\right\},$$

where m_i denotes the average of the degrees of the vertices adjacent to vertex v_i in G.

Lemma 2.7. [10] *Let G be a connected graph with $n \geq 3$ vertices. Then $\mu_2 \geq d_2$.*

Lemma 2.8. [7] *Let G be a connected graph with $n \geq 4$ vertices. Then $\mu_3 \geq d_3 - 1$.*

Lemma 2.9. [3] *If e is an edge of the graph G and $G' = G - e$. Then*

$$\mu_1(G) \geq \mu_1(G') \geq \mu_2(G) \geq \mu_2(G') \geq \cdots \geq \mu_{n-1}(G) \geq \mu_{n-1}(G') \geq \mu_n(G) = \mu_n(G') = 0.$$

3 Main results

First, some properties on the Laplacian eigenvalues of the double starlike tree $H(p, n, q)$ are stated as follows:

Lemma 3.1. *If $n \geq 4$ and $p > q \geq 2$, then the Laplacian eigenvalues of $H(p, n, q)$ satisfy*

(1) $p + 2 \leq \mu_1(H(p, n, q)) \leq p + 2 + \dfrac{1}{p + 2}$.

(2) $q + 2 \leq \mu_2(H(p, n, q)) \leq q + 3 + \dfrac{1}{q + 2}$.

(3) $\mu_3(H(p, n, q)) < 4$.

Proof. (1) It follows from Lemma 2.6 by a simple calculation.

(2) Let u and v be the vertices of degree $p + 1$ and $q + 1$ in $H(p, n, q)$, respectively. Then by Lemma 2.2, we have

$$\mu_1(H(p, n, q)) \geq \mu_1(H(p, n, q) - u) \geq \mu_2(H(p, n, q)) - 1.$$

Lemma 2.6 implies that

$$\mu_1(H(p, n, q) - u) \leq q + 2 + \frac{1}{q + 2}.$$

Then

$$\mu_2(H(p, n, q)) \leq \mu_1(H(p, n, q) - u) + 1 \leq q + 3 + \frac{1}{q + 2}.$$

Next, let G_1 be a subgraph of $H(p, n, q)$ obtained by deleting an edge whose end is neither u nor v. Clearly, G_1 has two connected components. Moreover, by Lemma 2.6, the largest Laplacian eigenvalue of each component is at least $q + 2$, *i.e,* $\mu_2(G_1) \geq q + 2$. Hence by Lemma 2.9 , $\mu_2(H(p, n, q)) \geq \mu_2(G_1) \geq q + 2$.

(3) Let M_{uv} be the $(p + n + q - 2) \times (p + n + q - 2)$ principal submatrix of $L(H(p, n, q))$ formed by deleting the rows and columns corresponding to u and v. Clearly, the largest eigenvalue of M_{uv} is less than 4. By Lemma 2.3, $\mu_3(H(p, n, q)) < 4$. □

Lemma 3.2. *Every graph L-cospectral to $H(p,n,q)$ with $n \geq 4$ and $p > q \geq 2$ is a tree with the degree sequence* $(p+1, q+1, \overbrace{2,\ldots,2}^{n-2}, \overbrace{1,\ldots,1}^{p+q})$.

Proof. Suppose that G' is L-cospectral to $H(p,n,q)$ with $n \geq 4$ and $p > q \geq 2$. By (1), (2), (4) and (5) of Lemma 2.5, G' is a tree with $n+p+q$ vertices and $n+p+q-1$ edges. Let $d_1 \geq d_2 \geq \cdots \geq d_{n+p+q}$ be the non-increasing degree sequence of graph G'. Now we determine the degree sequence of G'. By (1) of Lemma 3.1, we have $d_1 + 1 \leq \mu_1(G') \leq p+2+\dfrac{1}{p+2}$. Hence the largest degree $\Delta' = \Delta(G')$ of G' is at most $p+1$. Denote by n_i the number of vertices in G' with degree i, for $i = 1, 2, \ldots, \Delta' = d_1 \leq p+1$. By (1), (2) and (6) of Lemma 2.5, we have

$$\sum_{i=1}^{\Delta'} n_i = n+p+q, \tag{3.1}$$

$$\sum_{i=1}^{\Delta'} i n_i = 2(n+p+q-1), \tag{3.2}$$

$$\sum_{i=1}^{\Delta'} i^2 n_i = (p+1)^2 + (q+1)^2 + 4(n-2) + p + q. \tag{3.3}$$

By (3.1), (3.2) and (3.3), we have

$$\sum_{i=1}^{\Delta'} (i^2 - 3i + 2) n_i = p^2 + q^2 - p - q. \tag{3.4}$$

By Lemma 2.8 and (3) of Lemma 3.1, we have $d_3 \leq \mu_3(G') + 1 = \mu_3(H(p,n,q)) + 1 < 5$, which implies $d_3 \leq 4$. By Lemma 2.7 and (2) of Lemma 3.1, we have $d_2 \leq \mu_2(G') = \mu_2(H(p,n,q)) \leq q+3+\dfrac{1}{q+2}$, which implies $d_2 \leq q+3$. Lemma 2.1 implies that the line graphs of $\mathcal{L}(G')$ and $\mathcal{L}(H(p,n,q))$ are cospectral with respect to adjacency matrix, and by (3) of Lemma 2.5, they have the same number of triangles (sixth times of the number of closed walks of length 3), *i.e.*,

$$\sum_{i=1}^{\Delta'} \binom{i}{3} n_i = \binom{p+1}{3} + \binom{q+1}{3}. \tag{3.5}$$

Note that $\Delta'(= d_1) \leq p+1$, $d_2 \leq q+3$. We consider the following two cases.

Case 1. $q = 2$ or $q = 3$.

Case 1.1. $\Delta'(= d_1) < p+1$, *i.e.*, $n_{p+1} = 0$. By (3.5) and (3.4),

$$\binom{p+1}{3} + \binom{q+1}{3} = \sum_{i=1}^{\Delta'} \binom{i}{3} n_i \leq \frac{p}{6} \sum_{i=1}^{\Delta'} (i-1)(i-2) n_i = \frac{p}{6}(p^2 + q^2 - p - q).$$

If $q = 2$, we can get $p^2 - 3p + 6 \leq 0$, which is a contradiction to $p \geq 3$. If $q = 3$, we can get $p^2 - 7p + 24 \leq 0$, which is a contradiction to $p \geq 4$. Therefore $n_{p+1} \geq 1$.

Case 1.2. $n_{p+1} \geq 2$, *i.e.,* there are at least 2 vertices with the largest degree $\Delta' = p+1$. Then by (3.5),

$$\binom{p+1}{3} + \binom{q+1}{3} = \sum_{i=1}^{\Delta'} \binom{i}{3} n_i \geq 2\binom{p+1}{3} + \sum_{i=1}^{p} \binom{i}{3} n_i,$$

i.e.,

$$\binom{q+1}{3} - \binom{p+1}{3} \geq \sum_{i=1}^{p} \binom{i}{3} n_i.$$

This is a contradiction to $p > q \geq 2$. Hence $n_{p+1} = 1$. Further, if $q = 2$, by (3.5), we have $n_i = 0$ for $i = 4, \ldots, p$ and $n_3 = 1$. Moreover, by (3.1) and (3.2), we have $n_2 = n - 2$ and $n_1 = p + 2$. So the claim holds. If $q = 3$, by (3.5), we have $n_i = 0$ for $i = 5, \ldots, p$ and $n_4 \leq 1$. By (3.1), (3.2) and (3.3), it is easy to see that $n_4 = 1$. Moreover, by (3.1), (3.2) and (3.3), $n_3 = 0$, $n_2 = n - 2$ and $n_1 = p + 3$. So the claim holds.

Case 2. $q \geq 4$. Clearly, $d_1 \geq 4$ (Otherwise, if $d_1 = 3$, by (3.4), $n_3 = \frac{1}{2}(p^2 + q^2 - p - q)$, which is a contradiction to (3.5)). By (3.4) and (3.5), we have

$$\sum_{i=1}^{\Delta'} (i-1)(i-2)(i-3)n_i = p^3 + q^3 - 3p^2 - 3q^2 + 2p + 2q.$$

So,

$$6n_4 + \sum_{i=5}^{\Delta'} (i-1)(i-2)(i-3)n_i = p(p-1)(p-2) + q(q-1)(q-2). \tag{3.6}$$

Note that $d_3 \leq 4$, which implies that G' has at most two vertices of degree strictly greater than 4. Consider the following three cases.

Case 2.1. $d_1 = 4$, $d_2 \leq 4$. By (3.6),

$$6n_4 = p(p-1)(p-2) + q(q-1)(q-2).$$

By (3.1), (3.2) and (3.3), we have

$$n_3 = -\frac{1}{2}p(p-1)(p-3) - \frac{1}{2}q(q-1)(q-3) < 0,$$

a contradiction.

Case 2.2. $d_1 > 4$, $d_2 = 4$. By (3.6),

$$6n_4 = p(p-1)(p-2) + q(q-1)(q-2) - (d_1-1)(d_1-2)(d_1-3).$$

By (3.1), (3.2) and (3.3), we have

$$n_3 = \frac{1}{2}(d_1-1)(d_1-2)(d_1-4) - \frac{1}{2}p(p-1)(p-3) - \frac{1}{2}q(q-1)(q-3).$$

For $d_1 \le p+1$ and $q \ge 4$, $n_3 < 0$, a contradiction.

Case 2.3. $d_1 > 4$, $d_2 > 4$. By (3.6),

$$n_4 = \binom{p}{3} + \binom{q}{3} - \binom{d_1-1}{3} - \binom{d_2-1}{3}. \tag{3.7}$$

By (3.1), (3.2), (3.3) and (3.7), we have

$$n_3 = \frac{1}{2}(d_1-1)(d_1-2)(d_1-4) - \frac{1}{2}p(p-1)(p-3) + \frac{1}{2}(d_2-1)(d_2-2)(d_2-4) - \frac{1}{2}q(q-1)(q-3). \tag{3.8}$$

Case 2.3.1. $\Delta'(=d_1) < p+1$, *i.e.*, $n_{p+1} = 0$. Then consider the following cases.

Case 2.3.1.1. $\Delta'(=d_1) = p,\ d_2 = q+3$. By (3.7) and (3.8), we have

$$n_4 = \frac{1}{2}p^2 - \frac{3}{2}p + 1 - q^2,$$

and

$$n_3 = -\frac{3}{2}p^2 + \frac{11}{2}p - 5 + 3q^2 - 2q = -\frac{1}{2}(p-2)(3p-5) + 3q^2 - 2q.$$

Then $n_4 \ge 0$ implies that $p \ge \dfrac{3+\sqrt{8q^2+1}}{2}$. So,

$$n_3 \le -\frac{1}{2}\left(\frac{3+\sqrt{8q^2+1}}{2} - 2\right)\left(3 \times \frac{3+\sqrt{8q^2+1}}{2} - 5\right) + 3q^2 - 2q = -\frac{1}{2} + \frac{1}{2}\sqrt{8q^2+1} - 2q < 0,$$

a contradiction.

Case 2.3.1.2. $\Delta'(=d_1) = p-1,\ d_2 = q+3$. By (3.7) and (3.8), we have

$$n_4 = p^2 - 4p - q^2 + 4 = (p-2-q)(p-2+q),$$

and

$$n_3 = -3p^2 + 14p - 16 + 3q^2 - 2q = -(p-2)(3p-8) + 3q^2 - 2q.$$

Then $n_4 \ge 0$ implies that $p \ge q+2$. So,

$$n_3 \le -q(3q-2) + 3q^2 - 2q = 0.$$

Therefore $p = q+2$, and $n_3 = n_4 = 0$. But now $d_1 = p-1 = q+1 < d_2 = q+3$, a contradiction.

Case 2.3.1.3. $\Delta'(=d_1) \le p-2,\ d_2 = q+3$. Then $q+3 = d_2 \le d_1 \le p-2$ implies that $p \ge q+5$. By (3.8), we have

$$n_3 \le -\frac{9}{2}p^2 + \frac{51}{2}p - 37 + 3q^2 - 2q \le -\frac{9}{2}(q+5)^2 + \frac{51}{2}(q+5) - 37 + 3q^2 - 2q \le -\frac{3}{2}q^2 - \frac{43}{2}q - 22 < 0.$$

It is a contradiction.

Case 2.3.1.4. $\Delta'(=d_1) \le p$, $d_2 \le q+2$. By (3.8) and $p \ge q+1$, we have

$$n_3 \le -\frac{3}{2}p^2 + \frac{11}{2}p - 4 + \frac{3}{2}q^2 - \frac{5}{2}q = -\frac{1}{2}(p-1)(3p-8) + \frac{3}{2}q^2 - \frac{5}{2}q \le -\frac{1}{2}q(3q-5) + \frac{3}{2}q^2 - \frac{5}{2}q = 0.$$

Therefore $n_3 = 0$ implies that $d_1 = p$, $d_2 = q+2$ and $p = q+1$. But now $d_1 = p = q+1 < q+2 = d_2$, a contradiction.

Case 2.3.2. $\Delta'(=d_1) = p+1$, *i.e.,* $n_{p+1} \ge 1$. If $n_{p+1} \ge 2$, *i.e.,* there are at least 2 vertices with degree $\Delta' = p+1$, then by (3.5),

$$\binom{p+1}{3} + \binom{q+1}{3} = \sum_{i=1}^{\Delta'} \binom{i}{3} n_i \ge 2\binom{p+1}{3} + \sum_{i=1}^{p} \binom{i}{3} n_i,$$

i.e.,

$$\binom{q+1}{3} - \binom{p+1}{3} \ge \sum_{i=1}^{p} \binom{i}{3} n_i.$$

This is a contradiction to $p > q \ge 4$. Hence $n_{p+1} = 1$. For $q \ge 4$, by (3.7) and (3.8), it is easy to see that $d_2 = q+1$. By (3.5), $n_i = 0$ for $i = 3, 4, \ldots, q, q+2, \ldots, p$. By (3.1) and (3.2), $n_1 = p+q$ and $n_2 = n-2$. Therefore, the degree sequence of G' is $(p+1, q+1, \overbrace{2, \ldots, 2}^{n-2}, \overbrace{1, \ldots, 1}^{p+q})$. $\square$

Now we can present the main result of this paper.

Theorem 3.3. *Every double starlike tree $H(p, n, q)$ is determined by its Laplacian spectrum.*

Proof. If $n = 1$, $H(p, 1, q)$ is a star $K_{1,p+q}$, which is determined by its Laplacian spectrum [17]. If $n = 2$ or $n = 3$, $H(p, n, q)$ is determined by their Laplacian spectra [18]. If $p = q = 1$, $H(1, n, 1)$ is a path, which is determined by its Laplacian spectrum [4]. If $p > q = 1$, $H(p, n, 1)$ is a starlike tree, which is determined by its Laplacian spectrum [17]. If $p = q \ge 2$, $H(p, n, p)$ is determined by its Laplacian spectrum [11]. Hence we only need to consider $n \ge 4$, $p > q \ge 2$. Suppose that G' is L-cospectral to $H(p, n, q)$ with $n \ge 4$ and $p > q \ge 2$. By Lemma 3.2, G' is a tree with the degree sequence of G' is $(p+1, q+1, \overbrace{2, \ldots, 2}^{n-2}, \overbrace{1, \ldots, 1}^{p+q})$. In the following, we show that G' is isomorphic to $H(p, n, q)$.

First, we assume that the vertex of degree $p+1$ is adjacent to the vertex of degree $q+1$ and let a and b be such that there are $q-a$ (resp. $p-b$) vertices of degree 1 adjacent to the vertex of degree $q+1$ (resp. $p+1$) in G' (see Fig. 2); we have $0 \le a \le q$ and $0 \le b \le p$. Then,

$$\sum_{i=1}^{a} l'_i + \sum_{j=1}^{b} l''_j + (p+q) + 2 = n + p + q.$$

That is,

$$\sum_{i=1}^{a} l'_i + \sum_{j=1}^{b} l''_j = n - 2.$$

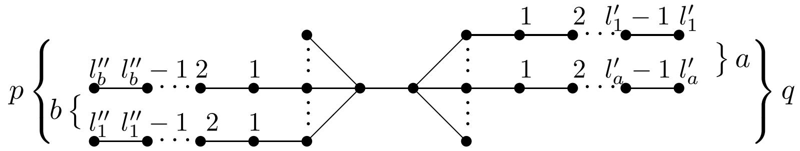

Fig. 2. The graph G'

For the line graph $\mathcal{L}(G')$, denote by n_i' the number of vertices with degree i. Then $n_1'=a+b$, $n_{p+q}'=1$, $n_{p+1}'=b$, $n_p'=p-b$, $n_{q+1}'=a$, $n_q'=q-a$, $n_2'=n+p+q-1-(n_1'+n_{p+q}'+n_{p+1}'+n_p'+n_{q+1}'+n_q')=n-2-a-b$, and $n_j'=0$, where $j\notin\{1,2,p,q,p+1,q+1,p+q\}$. On the other hand, it is easy to see that the degree sequence of line graph $\mathcal{L}(H(p,n,q))$ is $(p+1,q+1,\overbrace{p,\ldots,p}^{p},\overbrace{q,\ldots,q}^{q},\overbrace{2,\ldots,2}^{n-3})$. By (3) of Lemma 2.5, $\mathcal{L}(H(p,n,q))$ and $\mathcal{L}(G')$ have the same number of closed walks of length 4. Clearly, $\mathcal{L}(H(p,n,q))$ and $\mathcal{L}(G')$ have the same number of edges. Moreover, they have the same number of 4-cycles. Hence by Lemma 2.4, $\mathcal{L}(H(p,n,q))$ and $\mathcal{L}(G')$ have the same number of paths on three vertices. Therefore,

$$\begin{aligned}&\binom{p+1}{2}+\binom{q+1}{2}+p\binom{p}{2}+q\binom{q}{2}+(n-3)\binom{2}{2}\\ =&\binom{p+q}{2}+b\binom{p+1}{2}+(p-b)\binom{p}{2}+a\binom{q+1}{2}+(q-a)\binom{q}{2}+(n-2-a-b)\binom{2}{2}.\end{aligned}$$

Hence,

$$(p-1)(q-1)+b(p-1)+a(q-1)=0.$$

But for $p>q\geq 2$, $0\leq a\leq q$ and $0\leq b\leq p$, we have

$$(p-1)(q-1)+b(p-1)+a(q-1)\neq 0.$$

It is a contradiction.

Now assume that the vertex of degree $p+1$ is not adjacent to the vertex of degree $q+1$ and let a and b be such that there are $q-a$ (resp. $p-b$) vertices of degree 1 adjacent to the vertex of degree $q+1$ (resp. $p+1$) in G' (see Fig. 3); we have $0\leq a\leq q$ and $0\leq b\leq p$. Then,

$$l_1+\sum_{i=1}^{a}l_i'+\sum_{j=1}^{b}l_j''+(p+q)=n+p+q.$$

That is,

$$l_1+\sum_{i=1}^{a}l_i'+\sum_{j=1}^{b}l_j''=n. \tag{3.9}$$

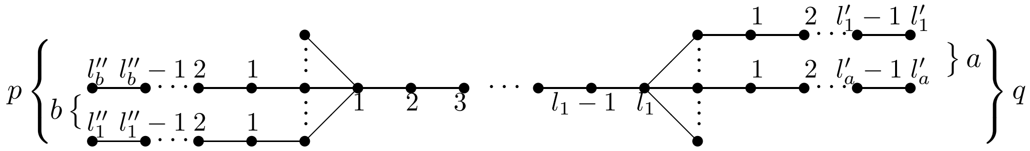

Fig. 3. The graph G'

For the line graph $\mathcal{L}(G')$, suppose that there exist n_i' vertices of degree i. Then $n_1'=a+b$, $n_{p+1}'=b+1$, $n_p'=p-b$, $n_{q+1}'=a+1$, $n_q'=q-a$, $n_2'=n+p+q-1-(n_1'+n_{p+1}'+n_p'+$

$n'_{q+1}+n'_q$)$=n-3-a-b$, and $n'_j=0$, where $j\notin\{1,2,p,q,p+1,q+1\}$. By (3) of Lemma 2.5, $\mathcal{L}(H(p,n,q))$ and $\mathcal{L}(G')$ have the same number of closed walks of length 4. Clearly, $\mathcal{L}(H(p,n,q))$ and $\mathcal{L}(G')$ have the same number of edges and the same number of 4-cycles. Hence by Lemma 2.4, $\mathcal{L}(H(p,n,q))$ and $\mathcal{L}(G')$ have the same number of paths on three vertices. Therefore,

$$\begin{aligned}&\binom{p+1}{2}+\binom{q+1}{2}+p\binom{p}{2}+q\binom{q}{2}+(n-3)\binom{2}{2}\\&=(b+1)\binom{p+1}{2}+(p-b)\binom{p}{2}+(a+1)\binom{q+1}{2}+(q-a)\binom{q}{2}+(n-3-a-b)\binom{2}{2}.\end{aligned}$$

Hence,

$$\frac{1}{2}p^3+\frac{1}{2}q^3+\frac{1}{2}p+\frac{1}{2}q+n-3=\frac{1}{2}p^3+\frac{1}{2}q^3+\frac{1}{2}p+\frac{1}{2}q+n-3+a(q-1)+b(p-1).$$

So $a(q-1)+b(p-1)$=0, *i.e.*, $a=0$ and $b=0$. By (3.9), we have $l_1=n$. Hence, all of the vertices of degree 1 are adjacent to the vertices of degree $p+1$ and the vertices of degree $q+1$ in the graph G'. Then G' is isomorphic to $H(p,n,q)$. □

Corollary 3.4. *The complement of a double starlike tree $H(p,n,q)$ is determined by its Laplacian spectrum.*

Proof. It follows from [8] and Theorem 3.3. □

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